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Benjamin–Feir type instability of Sine–Gordon equation and spectrum of Lamé equation

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Abstract

The Benjamin–Feir type instability of the nonhomoclinic wavetrain solution of the Sine–Gordon equation against the spatially periodic small perturbation is observed by relating the Fourier coefficients of the first order approximate solution of Fourier type to the band structure of the spectrum of the Lamé equation.

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1. Introduction

In this paper, we study the Benjamin–Feir type instability of the Sine–Gordon (SG) equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = 0. \tag{1}$$

In [2], Benjamin and Feir observed the behavior of the wavetrain of the nonlinear dispersive system against the small perturbation with the specific wavenumber, and they discovered the instability of the periodic Stokes wave on deep water. In connection with their pioneering work, the instability of this type is called now the *Benjamin–Feir type instability*.

Since the x-independent solutions $\phi_s(t,k)$, $k>0$ of the SG equation (1) solves the simple pendulum equation

$$\frac{d^2 \phi_s}{dt^2} + \sin \phi_s = 0 \tag{2}$$

there exist the following three x-independent solutions $\phi_s(t,k)$, $k>0$;

$$\phi_s(t,k) = \begin{cases} 2\sin^{-1}(k\,\mathrm{sn}(t,k)) & \text{if } 0<k<1, \\ 2\sin^{-1}(\mathrm{sn}(kt,k^{-1})) & \text{if } 1<k, \\ 4\tan^{-1}(e^{-t})+\pi & \text{if } k=1, \end{cases} \tag{3}$$

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where $\mathrm{sn}(t,k)$ is the Jacobi elliptic function with the modulus k. The purpose of the present work is to study the Benjamin–Feir type instability of the solutions $\phi_s(t,k),\ k>0$ against the small spatially periodic perturbation. By the way, the instability of the Benjamin–Feir type is a kind of the modulation instability. However, the last solution $\phi_s(t,1)$ is a homoclinic orbit, and there is no modulation wavenumber. So we omit it from our consideration. On the other hand, $\phi_s(t,k),\ k\neq 1$ are the nonhomoclinic orbits, so we call them the *nonhomoclinic solutions*. For the homoclinic case, in [1], the problem is considered by using Hirota's difference scheme.

The Benjamin–Feir zone $B(k)$ of the nonhomoclinic solution $\phi_s(t,k)$ is, roughly speaking, the set of the wavenumber of the spatially periodic perturbation of $\phi_s(t,k)$ such that the perturbed solution becomes unstable as $t\to\infty$. The purpose of the present work is to construct the Benjamin–Feir zone $B(k)$ for the nonhomoclinic solution $\phi_s(t,k),\ k\neq 1$. Moreover, we investigate the behavior of the perturbed solution at the boundary of $B(k)$. For the nonhomoclinic solution $\phi_s(t,k)$, we investigate the instability of those perturbed solutions by analyzing the band structure of the spectrum of Hill's operator

$$H_k=-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\cos\phi_s(t,k). \tag{4}$$

On the other hand, Bridges and Mielke [3] studied the Benjamin–Feir instability from quite universal standpoint of view. They formulated the linear stability problem for the Stokes periodic wavetrain on finitely deep water in terms of Hamiltonian structure of the water wave problem. However, it seems difficult to derive our results from them. The numerical study of the Benjamin–Feir instability will be reported in the paper [9].

The contents of this paper are as follows. In Section 2, the first order approximate solution of Fourier type is introduced and the problem is related to the spectral problem of Hill's operator H_k. In Section 3, the spectrum of Hill's operator introduced in Section 2 is computed exactly by our method. In Section 4, the Benjamin–Feir zones are obtained for the nonhomoclinic solutions mentioned above.

2. The first order approximate solution of Fourier type

Let $\phi_s(t,k),\ k\neq 1$ be the nonhomoclinic solution of the SG equation (1) defined by (3). Define the function $\phi_\varepsilon(x,t)$ by

$$\phi_\varepsilon(x,t)=\phi_s(t,k)+\varepsilon\sum_{n=0}^{\infty}\eta_n(t)\cos\mu_n x,\qquad \mu_n=\frac{2\pi n}{L}, \tag{5}$$

where ε is the small real parameter and L is a fixed positive constant. We assume that $\eta_n(t), n=0,1,2,\dots$ identically vanish with the finite number of exceptions. Put $\phi_\varepsilon(x,t)$ into the SG equation (1), then, by direct calculation, we have

$$\frac{\partial^2\phi_\varepsilon}{\partial t^2}-\frac{\partial^2\phi_\varepsilon}{\partial x^2}+\sin\phi_\varepsilon=\varepsilon\sum_{n=0}^{\infty}\left\{\frac{\mathrm{d}^2\eta_n}{\mathrm{d}t^2}+(\cos\phi_s)\eta_n+\mu_n^2\eta_n\right\}\cos\mu_n x+\mathrm{O}(\varepsilon^2).$$

Hence, if $\eta_n(t)$ solves the equation

$$\frac{\mathrm{d}^2\eta_n}{\mathrm{d}t^2}+(\cos\phi_s)\eta_n+\mu_n^2\eta_n=(\mu_n^2-H_k)\eta_n=0 \tag{6}$$

for each $n=0,1,2,\dots$, respectively, then

$$\frac{\partial^2\phi_\varepsilon}{\partial t^2}-\frac{\partial^2\phi_\varepsilon}{\partial x^2}+\sin\phi_\varepsilon=\mathrm{O}(\varepsilon^2)$$

follows. We call $\phi_\varepsilon(x,t)$ *the first order approximate solution of Fourier type* for the nonhomoclinic solution $\phi_s(t,k)$ if $\eta_n(t), n=0,1,2,\dots$ satisfy the eigenvalue problem (6).

Next, define $\phi_\varepsilon^{(1)}(x,t,\mu)$ by

$$\phi_\varepsilon^{(1)}(x,t,\mu)=\phi_s(t,k)+\varepsilon\eta(t)\,\cos\mu x,\qquad \mu>0, \tag{7}$$

where $\eta(t)$ is a solution of the eigenvalue problem

$$\frac{d^2\eta}{dt^2} + (\cos\phi_s)\eta + \mu^2\eta = (\mu^2 - H_k)\eta = 0. \tag{8}$$

That is, $\phi_\varepsilon^{(1)}(x,t,\mu)$ is a kind of the first order approximate solution of Fourier type such that only one component of the Fourier series (5) is nontrivial. We call $\phi_\varepsilon^{(1)}(x,t,\mu)$ the *single component perturbation*.

Now we define precisely the Benjamin–Feir zone $B(k)$ mentioned roughly in the Introduction.

Definition 1. Let $\phi_s(t,k)$ be the nonhomoclinic solution and $\phi_\varepsilon^{(1)}(x,t,\mu)$ be the corresponding single component perturbation. The subset $B(k)$ of the positive real axis $\boldsymbol{R}_+$ is called the Benjamin–Feir zone if the following condition is fulfilled; assume $(\eta(0),\eta'(0)) \neq (0,0)$, i.e. $\eta(t) \not\equiv 0$, then

$$\lim_{t\to\infty} \sup_{-\infty<x<\infty} |\phi_\varepsilon^{(1)}(x,t,\mu)| = \infty$$

holds if and only if $\mu \in B(k)$.

The purpose of the present work is firstly to obtain the Benjamin–Feir zone $B(k)$ and, secondly, to examine the asymptotic behavior of the first order approximate solution of Fourier type $\phi_\varepsilon(x,t)$ for large t when some of the wavenumbers $\mu_n,\ n=0,1,2,\ldots$ of the corresponding Fourier coefficients $\eta_n(t) \not\equiv 0,\ n=0,1,2,\ldots$ belong to $B(k)$.

We consider the first problem by converting the problem into the analysis of the band structure of the spectrum of the differential operator (4). To do so, by direct calculation, we have the following two kinds of potentials $U_k(t) = -\cos\phi_k(t)$;

$$U_k(t) = \begin{cases} 2k^2\,\mathrm{sn}^2(t,k) - 1, & 0<k<1, \\ 2\,\mathrm{sn}^2(kt,1/k) - 1, & k>1. \end{cases} \tag{9}$$

The differential operator H_k defined by (4) is expressed as

$$H_k = -\frac{d^2}{dt^2} + U_k(t).$$

Then the function $\eta_n(t),\ n=0,1,\ldots$ are the solutions of the eigenvalue problem

$$H_k\eta_n(t) = \mu_n^2\eta_n(t). \tag{10}$$

Hence, one can clarify the asymptotic behavior of the first order approximate solution of Fourier type $\phi_\varepsilon(x,t)$ as $t\to\infty$ by analyzing the behavior of the solution of the eigenvalue problem (10).

3. Band structure of the spectrum of Hill's operator

Since the potential $U_k(t)$ is a periodic one, the operator H_k is a kind of Hill's operator. In this section, before exact calculation of the spectrum of the operator H_k, we briefly review the generalities of the spectral theory of Hill's operator. We refer the reader to [5,6] for more precise information about the spectral theory of Hill's operator.

We consider the spectrum of the operator

$$H = -\frac{d^2}{dt^2} + V(t) \tag{11}$$

in the Hilbert space $L^2(-\infty,\infty)$, where $V(t)$ is the C^∞ real valued function with the period $l>0$. For this purpose, we consider the operator H defined by (11) in the class of the periodic functions with the period $2l$. Let $\theta_1(t,\lambda)$ and $\theta_2(t,\lambda)$ be the fundamental system of the solutions of the eigenvalue problem

$$H\theta(t,\lambda) = \lambda\theta(t,\lambda), \quad \lambda\in\boldsymbol{R} \tag{12}$$

with the initial condition

$$F(0,\lambda) = E_2, \tag{13}$$

where E_2 is the unit matrix of the size 2, and $F(t, \lambda)$ is the matrix defined by

$$F(t,\lambda)=\begin{pmatrix}\theta_1(t,\lambda) & \theta_2(t,\lambda)\\ \theta_1'(t,\lambda) & \theta_2'(t,\lambda)\end{pmatrix}.$$

The matrix $F(l, \lambda)$ is called the monodromy matrix, and

$$D(\lambda)=\operatorname{tr} F(l,\lambda)=\theta_1(l,\lambda)+\theta_2'(l,\lambda)$$

is called Hill's discriminant. We number the points such that $D(\lambda)=\pm 2$ as

$$-\infty=\lambda_{-1}<\lambda_0<\lambda_1\leqslant\lambda_2<\lambda_3\leqslant\lambda_4<\cdots<\lambda_{4j-1}\leqslant\lambda_{4j}<\lambda_{4j+1}\leqslant\lambda_{4j+2}<\cdots.$$

In the interval $I_i=(\lambda_{2i-1},\lambda_{2i}),\ i=0,1,2,\ldots,|D(\lambda)|>2$ holds. The intervals I_i are called the instability zones, and if $\lambda\in I_i$, then all nontrivial solutions of the eigenvalue problem (12) are unbounded, and grow exponentially as $t\to\infty$. On the one hand, in the intervals $J_i=(\lambda_{2i},\lambda_{2i+1}), i=0,1,2,\ldots,|D(\lambda)|<2$ holds. The intervals J_i is called the stability zone, and if $\lambda\in J_i$, then all solutions of the eigenvalue problem (12) are bounded, and are not periodic with the period l nor with the period $2l$. The points $\lambda_j,\ j=0,1,2,\ldots$ are periodic spectrum. If $\lambda_{2i-1}=\lambda_{2i}$, then the instability zone I_i degenerates to the empty set, and $\lambda_{2i-1}=\lambda_{2i}$ are the multiple periodic eigenvalues such that the dimension of the eigenspace is 2. On the other hand, if $\lambda_{2i-1}\neq\lambda_{2i}$ holds, then $\lambda_{2i-1}, \lambda_{2i}$ are simple periodic spectrum. In general, the instability zone are countably infinite. However, if they degenerate with the finite number of exceptions, they possesses the typical algebraic feature, and the potential $V(t)$ is called the finite zonal potential, or the algebro-geometric potential and are studied in detail.

To obtain the periodic spectrum, it suffices to solve the transcendental equation $D(\lambda)=\pm 2$. However, it is very difficult to obtain the exact values of them. But if we restrict the problem to obtain the simple periodic spectrum for the algebro-geometric potential, then the Λ-operator method is effective and the problem reduces to solve an algebraic equation. We refer the reader to [7,8,10] for the precise information about this method.

In what follows, we briefly explain the calculation of the simple periodic spectrum of the eigenvalue problem

$$H_k\eta=\mu^2\eta,$$

and the construction of the corresponding eigenfunctions by the Λ-operator method.

First we explain the case $0<k<1$.

I. *KdV polynomials*: The first three KdV polynomials are given by

$$Z_0(u)=1,\quad Z_1(u)=\tfrac{1}{2}u,\quad Z_2(u)=\tfrac{3}{8}u^2-\tfrac{1}{8}u''.$$

Put $u=U_k$ into them, and, by the direct calculation, we have

$$Z_2(U_k)=(k^4-\tfrac{1}{2}k^2)\operatorname{sn}^2(t,k)-\tfrac{1}{2}k^2+\tfrac{3}{8}.$$

II. *Express $Z_2(U_k)$ by the linear combination of $Z_0(U_k)$ and $Z_1(U_k)$*: We have

$$Z_2(U_k)=(k^2-\tfrac{1}{2})Z_1(U_k)+\tfrac{1}{8}Z_0(U_k).$$

Hence, the potential U_k turns out to be the first order algebro-geometric potential. Hence, the operator $H_k,\ \ 0<k<1$ has just three simple periodic eigenvalues $\lambda_1<\lambda_2<\lambda_3$. The instability zones are two open intervals $(-\infty,\lambda_1)$ and (λ_2,λ_3).

III. *Calculation of λ_j, $j=1,2,3$*: According to [7, p. 418, (12)], we construct the *M*-function $M_2(t,\lambda)$ (note that the notation $F(x,\lambda)$ is used in [7] for $M_2(t,\lambda)$);

$$M_2(t,\lambda)=k^2\operatorname{sn}^2(t,k)+\lambda-k^2. \tag{14}$$

Using this, we construct the spectral discriminant $\Delta(\lambda)$;

$$\begin{aligned}\Delta(\lambda)&=M_{2t}(t,\lambda)^2-2M_2(t,\lambda)M_2tt(t,\lambda)+4(U_k(t)-\lambda)M_2(t,\lambda)^2\\ &=-4\lambda(\lambda-k^2)(\lambda-k^2+1).\end{aligned}$$

Hence, the simple periodic spectrum of the operator H_k are $\lambda_1 = k^2 - 1$, $\lambda_2 = 0$, and $\lambda_3 = k^2$. The corresponding periodic eigenfunctions $\psi_j(t)$, $j = 1, 2, 3$ are given by

$$\psi_j(t) = \sqrt{M_2(t, \lambda_j)}, \quad j = 1, 2, 3.$$

Thus we have the following theorem.

Theorem 1. *Assume* $0 < k < 1$. *The simple periodic eigenvalues of the operator* H_k *are* $k^2 - 1$, 0, *and* k^2, *where* $k^2 - 1 < 0 < k^2$, *and the instability zones are the intervals* $(-\infty, k^2 - 1)$ *and* $(0, k^2)$.

Next we consider the case $k > 1$. In this case, by (9), we have

$$H_k\eta(t) = -\frac{d^2}{dt^2}\eta(t) + \left(2\,\mathrm{sn}^2\left(kt, \frac{1}{k}\right) - 1\right)\eta(t) = \mu^2\eta(t).$$

Let $\tau = kt$, then, by direct calculation, we have

$$-\frac{d^2}{d\tau^2}\tilde{\eta}(\tau) + \left(2\left(\frac{1}{k}\right)^2 \mathrm{sn}^2\left(\tau, \frac{1}{k}\right) - 1\right)\tilde{\eta}(\tau) = \tilde{\mu}^2\tilde{\eta}(\tau),$$

where

$$\tilde{\eta}(\tau) = \eta\left(\frac{\tau}{k}\right), \quad \tilde{\mu}^2 = \frac{1}{k^2} - 1 + \frac{\mu^2}{k^2}.$$

We write

$$\widetilde{H} = -\frac{d^2}{d\tau^2} + 2\left(\frac{1}{k}\right)^2 \mathrm{sn}^2\left(\tau, \frac{1}{k}\right) - 1.$$

Then, since $0 < 1/k < 1$, one can apply Theorem 1 to the operator $\widetilde{H}$, the simple periodic eigenvalues of the operator $\widetilde{H}$ are $1/k^2 - 1$, 0 and $1/k^2$, and the instability zones are the intervals $(-\infty, 1/k^2 - 1)$ and $(0, 1/k^2)$. If $\tilde{\mu}^2 = 1/k^2 - 1$ then $\mu^2 = 0$ follows. Similarly, if $\tilde{\mu}^2 = 0$, and $\tilde{\mu}^2 = 1/k^2$ then $\mu^2 = k^2 - 1$ and $\mu^2 = k^2$ follow, respectively. Hence, the instability zones of the operator H_k, $k > 1$ itself are two open intervals $(-\infty, 0)$ and $(k^2 - 1, k^2)$.

Theorem 2. *Assume* $1 < k$. *The simple periodic eigenvalues of the operator* H_k *are* 0, $k^2 - 1$ *and* k^2, *where* $0 < k^2 - 1 < k^2$, *and the instability zones are the intervals* $(-\infty, 0)$ *and* $(k^2 - 1, k^2)$.

4. Benjamin–Feir zone of Sine–Gordon equation

First suppose $0 < k < 1$. We consider the stability of the single component perturbation (7). If μ^2 belongs to the one of the instability zones of Hill's operator H_k, then, arbitrary nontrivial solution $\eta(t)$ of the eigenvalue problem (8) is unbounded, and grow exponentially as $t \to \infty$, which is mentioned in Section 3. Since the wavenumber μ is a positive real number, $\mu^2 > 0$ holds. Hence, by Theorem 1, the instability zone which includes μ^2 is the interval $(0, k^2)$. Therefore, the Benjamin–Feir zone $B(k)$ coincides with the interval $(0, k)$.

Similarly, one can see easily that if $1 < k$, the Benjamin–Feir zone $B(k)$ coincides with the interval $(\sqrt{k^2 - 1}, k)$.

Thus we have the following theorem concerned with the Benjamin–Feir zone and the asymptotic behavior of the single component perturbation $\phi_\varepsilon^{(1)}(x, t, \mu)$ as $t \to \infty$.

Theorem 3. *The Benjamin–Feir zones for* $0 < k$ *are given as follows.*

(1) *If* $0 < k < 1$, $B(k) = (0, k)$ *holds.*
(2) *If* $1 < k$, $B(k) = (\sqrt{k^2 - 1}, k)$ *holds.*

Moreover, if $\mu \in B(k)$, *then, the single component perturbation* $\phi_\varepsilon^{(1)}(x, t, \mu)$ *grows exponentially as* $t \to \infty$.

A part of this result itself seems to be already known. Actually, Ercolani et al. refer to this fact in [4] in the case $0<k<1$. However, the method of the proof is different from ours. Moreover, by the result of the present work, it seems that the methods developed for the soliton theory, which is nothing but a stable phenomenon, are quite useful even for a unstable phenomenon like Benjamin–Feir instability problem too.

Theorem 3 are concerned with the asymptotic behavior of the single component perturbation $\phi_\varepsilon^{(1)}(x,t,\mu)$ as $t\to\infty$. We apply those results to consider the asymptotic behavior of the first order approximate solution of Fourier type $\phi_\varepsilon(x,t)$ as $t\to\infty$.

Let S be the set of all wavenumbers μ_n such that $(\eta_n(0),\eta_n'(0))\neq(0,0)$, i.e., $\eta_n(t)\not\equiv 0$. Now suppose that

$$S\cap B(k)\neq\emptyset, \tag{15}$$

and $\mu_{n_0}\in S\cap B(k)$. Let $\eta_{n_0}(t)$ be the nontrivial eigenfunction of the eigenvalue problem

$$H_k\eta_{n_0}=\mu_{n_0}^2\eta_{n_0},$$

then $\eta_{n_0}(t)$ grows exponentially as $t\to\infty$. Moreover, since $\cos\mu_n x$, $n=0,1,2,\ldots$ are linearly independent, the function

$$G(x,t)=\sum_{\mu_n\in S\cap B(k)}\eta_n(t)\cos\mu_n x$$

grows exponentially as $t\to\infty$, and

$$\phi_\varepsilon(x,t)-G(x,t)=\phi_s(t,k)+\sum_{\mu_n\notin S\cap B(k)}\eta_n(t)\cos\mu_n x$$

is bounded as $t\to\infty$. Thus we have the following theorem.

Theorem 4. *For any $k\neq 1$, the first order approximate solution of Fourier type $\phi_\varepsilon(x,t)$ grows exponentially as $t\to\infty$, if and only if the condition* (15) *is fulfilled.*

Now let us consider the behavior of the single component perturbation at the boundary of $B(k)$ in the case $0<k<1$, i.e., $\mu=0$ and k.

First, we construct the real valued periodic eigenfunction $\theta_1(t,\lambda)$ for the eigenvalue $\lambda=0$ and k^2. As mentioned above, it is shown in [7, p. 420, Theorem 11] that the square root $M_2(t,\lambda)^{1/2}$ of the M-function $M_2(t,\lambda)$ mentioned above is the eigenfunction corresponding to the eigenvalue $\lambda=0$ and k^2. Hence, if $\lambda=0$, then we have

$$\theta_1(t,0)=\frac{-\mathrm{i}}{k}M_2(t,0)^{1/2}=\mathrm{cn}(t,k).$$

Since the dimension of the solution space of the eigenvalue problem (10) is 2, there is the solution $\theta_2(t,0)$ of (10) such that the pair of the solutions $\theta_1(t,0)$ and $\theta_2(t,0)$ is the fundamental system of solutions. The solution $\theta_2(t,0)$ is given by

$$\theta_2(t,0)=\mathrm{cn}(t,k)\int_0^t\frac{\mathrm{d}s}{\mathrm{cn}^2(s,k)}$$

which converges for $0\leqslant t\leqslant K$, where K is the constant defined by

$$K=\int_0^1\frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Note that these solutions satisfy the initial condition (13).

For $\lambda=k^2$, similarly to the above, we have

$$\theta_2(t,k^2)=\frac{1}{k}M_2(t,k^2)^{1/2}=\mathrm{sn}(t,k),\qquad\theta_1(t,k^2)=\mathrm{sn}(t,k)\int_t^K\frac{\mathrm{d}s}{\mathrm{sn}^2(s,k)}.$$

Note that these solutions satisfy the initial condition (13), too.

It is easy to see that $|\theta_2(t,0)|$, and$|\theta_1(t,k^2)|$ grow linearly, not exponentially, as $t \to \infty$. On the other hand, the asymptotic behavior of the single component perturbation $\phi_\varepsilon^{(1)}(x,t,\mu)$ at the boundary of $B(k)$ for $k>1$, i.e., $\mu=\sqrt{k^2-1}$ and k, is completely parallel to the above.

Therefore we have the following theorem.

Theorem 5. *Suppose* $\eta(0) \neq 0$, *then*

$$\lim_{t\to\infty} \sup_{-\infty<x<\infty} |\phi_\varepsilon^{(1)}(x,t,k)| = \infty$$

holds for all $k \neq 1$. *Moreover*, *suppose* $\eta'(0) \neq 0$, *then*

$$\lim_{t\to\infty} \sup_{-\infty<x<\infty} |\phi_\varepsilon^{(1)}(x,t,\mu)| = \infty$$

holds for $\mu=0$ *if* $0<k<1$, *and for* $\mu=\sqrt{k^2-1}$ *if* $k>1$.

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